A family of exactly-solvable driven-diffusive systems in one dimension

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Abstract

We introduce an exactly-solvable family of one-dimensional driven-diffusive systems defined on a discrete lattice. We find the quadratic algebra of this family which has an infinite-dimensional representation. We discuss the phase diagram of the system in a couple of special cases.

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One-dimensional driven-diffusive systems are systems of classical particles with hard-core interactions moving in a preferred direction which can be used to model many systems such as ribosomes moving along a m-RNA, ions diffusing in a narrow channel under the influence of an electric field or even cars proceeding on a long road [1–3]. These systems are usually defined on an open lattice coupled with two reservoirs at both ends or on a lattice with periodic boundary conditions. In the long-time limit the system settles into an non-equilibrium steady state characterized by some bulk density and the corresponding particle current. The out-of-equilibrium steady-state properties of these systems are not only affected by the boundaries but also to the localized inhomogeneities which play a crucial role not comparable to that in classical equilibrium systems. Driven-diffusive systems have been studied extensively in the past decade because of their unique non-equilibrium properties.

There are different approaches to study the steady state of these systems. The matrix product formalism (MPF) is one of the most powerful techniques in this field [4–6]. According to this formalism one assigns an operator to every state of a lattice site of the system. Now every configuration of the system is associated with a product of such operators. The steady-state probability of such configuration is then given by a trace (for the systems defined on a ring geometry) or a matrix element (for the systems defined on an open lattice) of such products. For instance for a three-states system on a lattice with the periodic boundary condition we define three operators, let us say A, B and E, associated with three different states of each lattice site. The steady-state weight of a given configuration similar to $ABEEABEAA$ is then proportional to $\text{Tr}[ABEEABEAA]$. Requiring that the probability distribution function of the system defined above is stationary provides us with an algebra of operators. For the systems with nearest-neighbors interactions it turns out that the algebra is quadratic. In order

to calculate the steady-state weights one can either work with the commutation relation of the operators or find a representation for the associated quadratic algebra of the system (for a recent review on the MPF, see [4]).

So far only a couple of one-dimensional driven-diffusive models have been solved exactly using the MPF [4]. Therefore, it would be interesting to investigate whether or not there are other models which can be solved exactly using this technique. In this paper, we introduce a large family of one-dimensional driven-diffusive models which under some constraints on the reaction rates can be solved exactly, i.e. the steady-state weight of any configuration can be calculated rigorously using the MPF. From there the mean values of physical quantities such as the currents of particles and also their concentrations can be obtained exactly. This three-states family belongs to the systems with non-conserving dynamics and nearest-neighbors interactions. These states belong to two different types of particles and holes on each lattice site. It is not clear that the steady-state weights of such a system can generally be written as matrix product states; however, we will present a constraint under which it will be possible to write these weights using the MPF. In this regard, we will relate the quadratic algebra of our system to the quadratic algebra of the systems with known representations. A couple of members of this family of systems have already been studied in literature. At the end of this paper, we will briefly discuss some other special cases which have not been studied before.

During last decade several quadratic algebras have been introduced and used to study the critical behaviors of one-dimensional driven-diffusive systems. It has also been tried to classify certain quadratic algebras and find their representations [7–9]. One of the most well-known quadratic algebras belongs to the partially asymmetric simple exclusion process (PASEP) in which the classical particles hop to the left and to the right with the rates p and q on an open one-dimensional lattice of the length L. The jumps of particles are only successful provided that the target sites are empty. The particles are injected into the system from the left boundary with the rate α provided that the first site of the lattice is empty. The particles are also extracted from the last site of the lattice provided that it is already occupied. The particles are subjected to the hardcore interactions so that two particles cannot occupy a single site of the lattice simultaneously. It is known that the steady-state weights of the PASEP can be written in terms of products of infinite-dimensional square matrices which satisfy a quadratic algebra of the following form [5, 10, 11]:

$$pAB - qBA = A + B$$
$$\alpha A|V\rangle = |V\rangle \tag{1}$$
$$\beta\langle W|B = \langle W|$$

in which the operators A and B are associated with two different states of each lattice site, i.e. a particle and an empty site respectively. It is known that this algebra has an infinite-dimensional matrix representation for $p \geqslant q$ given by the following matrices [5]

$$A = \frac{1}{p-q} \begin{pmatrix} 1-a & d_1 & 0 & 0 & \cdots \\ 0 & 1-a\left(\dfrac{q}{p}\right) & d_2 & 0 & \\ 0 & 0 & 1-a\left(\dfrac{q}{p}\right)^2 & d_3 & \\ 0 & 0 & 0 & 1-a\left(\dfrac{q}{p}\right)^3 & \\ \vdots & & & & \ddots \end{pmatrix}$$

$$B = \frac{1}{p-q}\begin{pmatrix} 1-b & 0 & 0 & 0 & \cdots \\ d_1 & 1-b\left(\dfrac{q}{p}\right) & 0 & 0 & \\ 0 & d_2 & 1-b\left(\dfrac{q}{p}\right)^2 & 0 & \\ 0 & 0 & d_3 & 1-b\left(\dfrac{q}{p}\right)^3 & \\ \vdots & & & & \ddots \end{pmatrix}$$

(2)

and vectors

$$|V\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \qquad \langle W| = (1 \quad 0 \quad 0 \quad 0 \quad \cdots)$$

(3)

in which we have defined $a = 1 - \frac{p-q}{\alpha}$, $b = 1 - \frac{p-q}{\beta}$ and $d_i{}^2 = \left(1 - \left(\frac{q}{p}\right)^i\right)\left(1 - ab\left(\frac{q}{p}\right)^{i-1}\right)$. The matrix representation of (1) given by (2) and (3) is quite well defined in a sense that the product of any number of these matrices sandwiched between $\langle W|$ and $|V\rangle$ is finite. In what follows we will look for those systems which their steady states in terms of the MPF are given by (1) and its representation, i.e. (2) and (3).

Let us define a new operator $E = \omega|V\rangle\langle W|$ in which ω is a real number. By multiplying $\langle W|\omega$ from the right in the second row and also $\omega|V\rangle$ from the left in the third row of (1) one finds

$$pAB - qBA = A + B$$
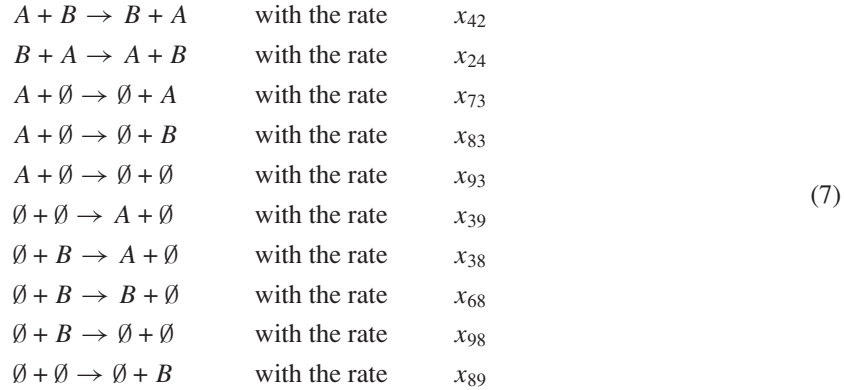$$\alpha AE = E \tag{4}$$
$$\beta EB = E,$$

which was first introduced in [12]. Obviously, this algebra has the same representation given by (2) and (3) and one should only define E as

$$E = \omega|V\rangle\langle W| = \begin{pmatrix} \omega & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ \vdots & & & & \ddots \end{pmatrix}.$$

(5)

It can easily be verified that the operator E has the property $E^2 = \omega E$ and also can be added to (4) without changing its matrix representation to make a larger and well-defined quadratic algebra. Therefore, the following quadratic algebra has the same matrix representation given by (2), (3) and (5)

$$pAB - qBA = A + B$$
$$\alpha AE = E$$
$$\beta EB = E \tag{6}$$
$$E^2 = \omega E.$$

The question is now whether or not one can find a three-states model, defined on a lattice with a ring geometry, with a steady state described by (6). By applying the standard MPF [4, 6] we have found that the steady-state probability distribution function of the following three-states system:

$$
\begin{aligned}
A + B &\rightarrow B + A &\text{with the rate} &\quad x_{42} \\
B + A &\rightarrow A + B &\text{with the rate} &\quad x_{24} \\
A + \emptyset &\rightarrow \emptyset + A &\text{with the rate} &\quad x_{73} \\
A + \emptyset &\rightarrow \emptyset + B &\text{with the rate} &\quad x_{83} \\
A + \emptyset &\rightarrow \emptyset + \emptyset &\text{with the rate} &\quad x_{93} \\
\emptyset + \emptyset &\rightarrow A + \emptyset &\text{with the rate} &\quad x_{39} \\
\emptyset + B &\rightarrow A + \emptyset &\text{with the rate} &\quad x_{38} \\
\emptyset + B &\rightarrow B + \emptyset &\text{with the rate} &\quad x_{68} \\
\emptyset + B &\rightarrow \emptyset + \emptyset &\text{with the rate} &\quad x_{98} \\
\emptyset + \emptyset &\rightarrow \emptyset + B &\text{with the rate} &\quad x_{89}
\end{aligned}
\tag{7}
$$

in which A, B and E can be associated with two particles of different types and an empty site respectively, can be described by (6) provided that we define

$$
\begin{aligned}
p &= x_{42} \\
q &= x_{24} \\
\alpha &= x_{73} \\
\beta &= x_{68} \\
\omega &= \frac{1}{x_{39} + x_{89}} \left(\frac{x_{93}}{x_{73}} + \frac{x_{98}}{x_{68}} \right)
\end{aligned}
\tag{8}
$$

and require the parameters to satisfy the following constraint:

$$
x_{39} \left(\frac{x_{83}}{x_{73}} - \frac{x_{38} + x_{98}}{x_{68}} \right) = x_{89} \left(\frac{x_{38}}{x_{68}} - \frac{x_{93} + x_{83}}{x_{73}} \right).
\tag{9}
$$

This constraint together with definitions (8) guarantee that our model defined by (7) has a well-defined algebra given by (6) and that its steady-state weights can be expressed in terms of a matrix product states.

　　As we mentioned, a couple of special cases of this family of models have already been studied in literature. For instance in [13] the authors have studied an exactly solvable model with the constraints $x_{83} = x_{38} = 0$, $x_{42} = \frac{1}{q} x_{24} = x_{73} = x_{68} = x_{39} = x_{89} = 1$ and $x_{93} = x_{98} = \omega$. In this case, the particle number for both A and B particles changes because of creation and annihilation of them. They have found that the system undergoes a continuous phase transition from a fluid phase to a maximal current phase for $q < 1$ by varying ω. For $q > 1$ the system is always phase separated and there is no phase transition. In another paper, the authors have considered a special case in which $x_{24} = x_{83} = x_{38} = x_{98} = x_{89} = 0$, $x_{42} = x_{68} = x_{73} = x_{39} = 1$ and $x_{93} = \omega$ [14]. In this case, the number of B particles is conserved while the number of A particles can change due to the creation and annihilation. It has been shown that, in this case where a finite density of B particles exists on the ring, the model can be solved exactly and undergoes a second-order phase transition by varying ω from a phase in which the density of the empty sites is zero to another phase where the density of the empty sites is nonzero. In [15], the authors have studied the same process with more general reaction rates as $x_{24} = x_{83} = x_{38} = x_{98} = x_{89} = 0$, $x_{42} = \frac{1}{\beta} x_{68} = \frac{1}{\alpha} x_{73} = x_{39} = 1$ and $x_{93} = \omega$; however, in this case they assume that there is only a single B particle in the

system. They have found that the phase transition is now discontinuous and that shocks might appear in the system at the transition point.

In what follows we consider the most general case where most of the parameters in (7) are nonzero which has not been studied in literature yet. We first study the phase diagram of the model for $x_{24} = 0$ and $x_{42} = 1$.[1] The partition function of the system defined as the sum of the weights of all accessible configurations with at least one empty site, is given by

$$Z_L(\alpha, \beta, \omega, \xi) = Tr[(\xi A + B + E)^L] - Tr[(\xi A + B)^L]. \tag{10}$$

The fugacity of A particles ξ is an auxiliary parameter and the reason we have defined it will be clear shortly. It turns out that the generating function of this partition function can be calculated exactly. After some straightforward calculations one finds

$$G(\alpha, \beta, \omega, \xi, \lambda) = \sum_{L=1}^{\infty} \lambda^L Z_L = \frac{\omega \lambda \frac{\partial}{\partial \lambda} U}{1 - \omega U} \tag{11}$$

in which we have defined

$$U(\alpha, \beta, \xi, \lambda) = \sum_{L=0}^{\infty} \lambda^{L+1} \langle W|(\xi A + B)^L|V\rangle.$$

It turns out that

$$U(\alpha, \beta, \xi, \lambda) = \frac{4\lambda}{f^-(\alpha) f^+(\beta)}, \tag{12}$$

where

$$f^{\pm}(x) = \frac{1}{x}\left(1 - 2x \pm \lambda(1 - \xi) - \sqrt{(1 + \lambda(1 - \xi))^2 - 4\lambda}\right).$$

The phase diagram of the system can now be obtained by studying the singularities of the generating function (11) for $\xi = 1$. One can easily see that in this case the generating function has two different kinds of singularities: a square root singularity $\lambda^* = \frac{1}{4}$ and two simple pole singularities which come from denominator of (11) by solving the equation $1 - \omega U(\alpha, \beta, \xi = 1, \lambda^*) = 0$. However, analyzing the absolute values of singularities shows that the system can only have two phases: a maximal current phase which is specified by the square root singularity (for $\alpha + \beta > 1$ and $\left(2 - \frac{1}{\alpha}\right)\left(2 - \frac{1}{\beta}\right) \geqslant \omega$) and a fluid phase which is specified by the simple pole singularity (for $\alpha + \beta < 1$ or $\alpha + \beta > 1$ and $\left(2 - \frac{1}{\alpha}\right)\left(2 - \frac{1}{\beta}\right) < \omega$). Keeping $\xi = 1$ the total density of the empty sites ρ_E can easily be calculated using

$$\rho_E = \lim_{L \to \infty} \frac{\omega}{L} \frac{\partial}{\partial \omega} \ln Z_L. \tag{13}$$

In the thermodynamic limit we have $\rho_E \sim -\omega \frac{\partial \ln \lambda^*}{\partial \omega}$. It turns out that the density of the empty sites is zero in the maximal current phase while it is nonzero in the fluid phase. The total density of A and B particles in each phase can also be calculated exactly. Since the density of the empty sites is known only one of these densities is independent. The auxiliary fugacity ξ can now help us to find the density of A particles as

$$\rho_A = \lim_{L \to \infty} \frac{\xi}{L} \frac{\partial}{\partial \xi} \ln Z_L|_{\xi=1}. \tag{14}$$

The density of the B particles is then $\rho_B = 1 - \rho_E - \rho_A$. We have found that in the maximal current phase $\rho_A = \rho_B = \frac{1}{2}$. In the fluid phase both ρ_A and ρ_B are complicated functions of α, β and ω and will not be presented here. The particle currents for both species can also

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[1] By resealing the time one can always take one of the parameters equal to 1.

be calculated exactly in this case. We have found that the current of A particles J_A is always equal to that of B particles J_B. Our calculations also show that the particle current is given by $J_A = J_B = \frac{Z_{L-1}}{Z_L}$ which is equal to λ^* in the thermodynamic limit. In the maximal current phase we simply find them to be equal to $\frac{1}{4}$. This is actually in contrast with the case studied in [14] where the number of B particles in one species is conserved. It has been shown that in this case the currents can be different. For $x_{24} = q$, $x_{42} = 1$, $\alpha = 1$ and $\beta = 1$ the results are exactly those obtained in [13]. On a lattice with periodic boundary conditions we have found that the phase diagram of the model does not change even for arbitrary α and β; therefore, we will not discuss this case here.

One should note that (6) can also explain the steady state of a system with open boundaries and two species of particles. The particles of type A (B) are injected from the left (right) boundary with the rate $\frac{1}{\omega}$ ($\frac{1}{\omega}$) and extracted from the right (left) boundary with the rate α (β). All of the processes in (7) might also take place on the lattice. The partition function of the model for $x_{24} = 0$ and $x_{42} = 1$ can also be calculated exactly and is given by

$$Z_L(\alpha, \beta, \omega) = \langle W|(A + B + E)^L|V\rangle = \sum_{i=1}^{L} \frac{i(2L - i - 1)!}{L!(L - i)!} \frac{\tilde{\alpha}^{-i-1} - \tilde{\beta}^{-i-1}}{\tilde{\alpha}^{-1} - \tilde{\beta}^{-1}} \tag{15}$$

in which

$$\tilde{\alpha} = \frac{\alpha}{1 + \alpha\omega\lambda_1}, \qquad \tilde{\beta} = \frac{\beta}{1 + \beta\omega\lambda_2} \tag{16}$$

and

$$\lambda_1 = \frac{1}{2\alpha\beta\omega}\left(\alpha - \beta + \alpha\beta\omega - \sqrt{(\alpha - \beta + \alpha\beta\omega)^2 + 4\alpha\beta^2\omega(1 - \alpha)}\right), \tag{17}$$

$$\lambda_2 = \frac{-1}{2\alpha\beta\omega}\left(\alpha - \beta - \alpha\beta\omega - \sqrt{(\alpha - \beta + \alpha\beta\omega)^2 + 4\alpha\beta^2\omega(1 - \alpha)}\right). \tag{18}$$

The phase diagram structure of the model can now be obtained by studying the thermodynamic behavior of the partition function. Equivalently, one can study the zeros of this partition function as a function of α, β or ω. It turns out that the system has again two different phases. The phase transition occurs for $\alpha + \beta > 1$ at $\omega = \left(2 - \frac{1}{\alpha}\right)\left(2 - \frac{1}{\beta}\right)$ similar to the ring geometry case. Both phases are symmetric, that is the currents of particles of different types are always equal.

In this paper, we have investigated a general quadratic algebra (6) associated with a family of exactly solvable three-states reaction-diffusion systems with non-conserving dynamics defined on a one-dimensional lattice with a ring geometry. This algebra is in fact the quadratic algebra associated with the PASEP given by (1) besides the relation $E^2 = \omega E$ which does not change the representation of the algebra but it generates a new algebra which allows us to study a new family of three-states processes using the MPF. This family of three-states processes is defined by ten nonzero reaction rates given by (7) which should satisfy a constraint given by (9). Under this constraint the steady state of the system can be written as a matrix product form. We have considered the most general model of this type on a lattice with periodic boundary conditions and studied its phase diagram. As we have also mentioned, the generalized algebra (6) can explain the steady state of a three-states system with the reactions defined by (7) but this time under open boundary conditions where the particles are allowed to enter and leave the lattice with some specific injection and extraction rates. The phase diagram of the model has also been studied and the partition function calculated exactly. Our approach can be generalized and applied to other models similar to the model studied in [16] (which is a p-species model defined on a lattice with periodic boundary conditions) to find a $(p+1)$-species exactly solvable model. The results will be published elsewhere.

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